Subtract first equation: $5y = 20$

$$y = 4$$
$$x + 4 = 8$$
$$x = 4$$

So they intersect at (4, 4)

16c $5x - 7y = 3$ and $2x + 8y = 3$

Multiply first equation by 2:
$10x - 14y = 6$ (1)
Multiply second equation by 5:
$10x + 40y = 15$ (2)

$(2) - (1): 54y = 9$

$$y = \frac{1}{6}$$

Substitute y value into either of the original equations:

$$2x + 8\left(\frac{1}{6}\right) = 3$$
$$2x = \frac{5}{3}$$
$$x = \frac{5}{6}$$

So they intersect at $\left(\frac{5}{6}, \frac{1}{6}\right)$

16d $-8x + 5y = 1$ and $3x + 18y + 7 = 0$

Multiply first equation by 3:
$-24x + 15y = 3$ (1)
Rearrange and multiply second equation by 8: $24x + 144y = -56$ (2)

$(1) + (2): 159y = -53$

$$y = -\frac{1}{3}$$

Substitute y value into either of the original equations:

$$3x + 18\left(-\frac{1}{3}\right) + 7 = 0$$
$$3x = -1$$
$$x = -\frac{1}{3}$$

So they intersect at $\left(-\frac{1}{3}, -\frac{1}{3}\right)$

Try it 1G

1a $(x+2)^2 + (y-8)^2 = 25$

$a = -2$, $b = 8 \Rightarrow$ centre is $(-2, 8)$
Radius is $\sqrt{25} = 5$

1b $a = 7$, $b = -9$, $r = 8$ so equation is
$(x-7)^2 + (y-(-9))^2 = 8^2$

$$(x-7)^2 + (y+9)^2 = 64$$

2a $x^2 + y^2 - 10y + 16 = 0$

Complete the square for $y^2 - 10y$:

$$x^2 + (y-5)^2 - 25 + 16 = 0$$
$$x^2 + (y-5)^2 = 9$$

Centre is (0, 5) and radius is $\sqrt{9} = 3$

2b $x^2 + y^2 + 6x - 12y = 0$

Group x terms and y terms:

$x^2 + 6x + y^2 - 12y = 0$

Complete the square:

$$(x+3)^2 - 9 + (y-6)^2 - 36 = 0$$
$$(x+3)^2 + (y-6)^2 = 45$$

Centre is $(-3, 6)$ and radius is $\sqrt{45} = 3\sqrt{5}$

3 Centre is $\left(\frac{4+2}{2}, \frac{6+-4}{2}\right) = (3, 1)$

Radius is: $r = \frac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$\frac{1}{2}\sqrt{(2-4)^2 + (-4-6)^2}$$
$$= \frac{1}{2}\sqrt{(-2)^2 + (-10)^2}$$
$$= \sqrt{26}$$
$$(x-3)^2 + (y-1)^2 = 26$$

4a $(x-1)^2 + (y+4)^2 = 50$

Substitute $x = 6$, $y = 1$:

$(6-1)^2 + (1+4)^2 = 5^2 + 5^2$
$= 50$ so (6, 1) lies on the circle

4b Centre is $(1, -4)$ so gradient of radius to (6, 1) is: $\frac{1--4}{6-1} = \frac{5}{5}$
$= 1$

The gradient of the tangent is $m = -1$ since $-1 \times 1 = -1$

Equation of tangent is $y - 1 = -1(x - 6)$

$y = -x + 7$

5a Rearrange $3x + y = 5$:

$$y = 5 - 3x$$

$$x^2 + (5 - 3x - 4)^2 = 17$$
$$x^2 + (1 - 3x)^2 = 17$$
$$x^2 + 1 - 6x + 9x^2 = 17$$
$$10x^2 - 6x - 16 = 0$$
$$x = 1.6, -1$$

Substitute x values into $y = 5 - 3x$:

$$y = 5 - 3(1.6)$$
$$= 0.2$$
$$y = 5 - 3(-1)$$
$$= 8$$

So they intersect at $A(1.6, 0.2)$ and $B(-1, 8)$

5b Length of chord $AB = \sqrt{(-1-1.6)^2 + (8-0.2)^2}$

$$= \sqrt{(-2.6)^2 + 7.8^2}$$
$$= \frac{13}{5}\sqrt{10} \text{ (= 8.22 to 3sf)}$$

6 Rearrange $2x - y + 11 = 0$:

$y = 2x + 11$

Substitute into the equation of the circle:

$$(x-5)^2 + (2x + 11 - 1)^2 = 80$$
$$(x-5)^2 + (2x + 10)^2 = 80$$
$$x^2 - 10x + 25 + 4x^2 + 40x + 100 = 80$$
$$5x^2 + 30x + 45 = 0$$

$$b^2 - 4ac = 30^2 - 4 \times 5 \times 45$$
$= 0$ so exactly one solution

Therefore the line and the circle touch once, hence the line is a tangent to the circle.

Bridging Exercise 1G

1a $(x-2)^2 + (y-5)^2 = 49$

1b $(x+1)^2 + (y+3)^2 = 16$

1c $(x+3)^2 + y^2 = 2$

1d $(x-4)^2 + (y+2)^2 = 5$

2a Centre is (5, 3), radius is $\sqrt{16} = 4$

2b Centre is (–3, 4), radius is $\sqrt{36} = 6$

2c Centre is (9, –2), radius is $\sqrt{100} = 10$

2d Centre is (–3, –1), radius is $\sqrt{80} = 4\sqrt{5}$

2e Centre is $(\sqrt{2}, -2\sqrt{2})$, radius is $\sqrt{32} = 4\sqrt{2}$

2f Centre is $\left(-\frac{1}{4}, -\frac{1}{3}\right)$, radius is $\sqrt{\frac{25}{4}} = \frac{5}{2}$

3a $x^2 + 2x + y^2 = 24$

$$(x+1)^2 - 1 + y^2 = 24$$
$$(x+1)^2 + y^2 = 25$$

Centre is (–1, 0), radius is $\sqrt{25} = 5$

3b $x^2 + y^2 + 12y = 13$

$$x^2 + (y+6)^2 - 36 = 13$$
$$x^2 + (y+6)^2 = 49$$

Centre is (0, –6), radius is $\sqrt{49} = 7$

3c $x^2 + y^2 - 4x + 3 = 0$

$$x^2 - 4x + y^2 + 3 = 0$$
$$(x-2)^2 - 4 + y^2 + 3 = 0$$
$$(x-2)^2 + y^2 = 1$$

Centre is (2, 0), radius is $\sqrt{1} = 1$

3d $x^2 + y^2 + 6x + 8y + 2 = 0$

$$x^2 + 6x + y^2 + 8y + 2 = 0$$
$$(x+3)^2 - 9 + (y+4)^2 - 16 + 2 = 0$$
$$(x+3)^2 + (y+4)^2 = 23$$

Centre is (–3, –4), radius is $\sqrt{23}$

3e $x^2 + y^2 - 8x - 10y = 3$

$$x^2 - 8x + y^2 - 10y = 3$$
$$(x-4)^2 - 16 + (y-5)^2 - 25 = 3$$
$$(x-4)^2 + (y-5)^2 = 44$$

Centre is (4, 5), radius is $\sqrt{44} = 2\sqrt{11}$

3f $x^2 + y^2 + 14x - 2y = 5$

$$x^2 + 14x + y^2 - 2y = 5$$
$$(x+7)^2 - 49 + (y-1)^2 - 1 = 5$$
$$(x+7)^2 + (y-1)^2 = 55$$

Centre is (–7, 1), radius is $\sqrt{55}$

3g $x^2 + y^2 + 5x - 4y + 3 = 0$

$$x^2 + 5x + y^2 - 4y + 3 = 0$$
$$\left(x + \frac{5}{2}\right)^2 - \frac{25}{4} + (y-2)^2 - 4 + 3 = 0$$
$$\left(x + \frac{5}{2}\right)^2 + (y-4)^2 = \frac{29}{4}$$

Centre is $\left(-\frac{5}{2}, 2\right)$, radius is $\sqrt{\frac{29}{4}} = \frac{1}{2}\sqrt{29}$

3h $x^2+y^2-3x-9y=2$

$x^2-3x+y^2-9y=2$

$\left(x-\frac{3}{2}\right)^2-\frac{9}{4}+\left(y-\frac{9}{2}\right)^2-\frac{81}{4}=2$

$\left(x-\frac{3}{2}\right)^2+\left(y-\frac{9}{2}\right)^2=\frac{49}{2}$

Centre is $\left(\frac{3}{2},\frac{9}{2}\right)$, radius is $\sqrt{\frac{49}{2}}=\frac{7}{2}\sqrt{2}$

3i $x^2+y^2-x+7y+12=0$

$x^2-x+y^2+7y+12=0$

$\left(x-\frac{1}{2}\right)^2-\frac{1}{4}+\left(y+\frac{7}{2}\right)^2-\frac{49}{4}+12=0$

$\left(x-\frac{1}{2}\right)^2+\left(y+\frac{7}{2}\right)^2=\frac{1}{2}$

Centre is $\left(\frac{1}{2},-\frac{7}{2}\right)$, radius is $\sqrt{\frac{1}{2}}=\frac{\sqrt{2}}{2}$

4a Midpoint is $\left(\frac{3+1}{2},\frac{5+7}{2}\right)=(2,\ 6)$

Radius is $\frac{1}{2}\sqrt{(1-3)^2+(7-5)^2}$

$=\frac{1}{2}\sqrt{(-2)^2+2^2}$

$=\sqrt{2}$

So equation is

$(x-2)^2+(y-6)^2=(\sqrt{2})^2$

$(x-2)^2+(y-6)^2=2$

4b Midpoint is $\left(\frac{4+2}{2},\frac{-1+(-5)}{2}\right)=(3,\ -3)$

Radius is $\frac{1}{2}\sqrt{(2-4)^2+(-5--1)^2}$

$=\frac{1}{2}\sqrt{(-2)^2+(-4)^2}$

$=\sqrt{5}$

So equation is

$(x-3)^2+(y-(-3))^2=(\sqrt{5})^2$

$(x-3)^2+(y+3)^2=5$

4c Midpoint is $\left(\frac{1+-9}{2}+\frac{-3+-6}{2}\right)=(-4,\ -4.5)$

Radius is $\frac{1}{2}\sqrt{(-9-1)^2+(-6-(-3))^2}$

$=\frac{1}{2}\sqrt{(-10)^2+(-3)^2}$

$=\frac{1}{2}\sqrt{109}$

So equation is

$(x-(-4))^2+(y-(-4.5))^2=\left(\frac{1}{2}\sqrt{109}\right)^2$

$(x+4)^2+(y+4.5)^2=27.25$

4d Midpoint is

$\left(\frac{-3+8}{2},\frac{-7+(-16)}{2}\right)=(2.5,\ -11.5)$

Radius is $\frac{1}{2}\sqrt{(8-(-3))^2+(-16-(-7))^2}$

$=\frac{1}{2}\sqrt{11^2+(-9)^2}$

$=\frac{1}{2}\sqrt{202}$

So equation is

$(x-2.5)^2+(y-(-11.5))^2=\left(\frac{1}{2}\sqrt{202}\right)^2$

$(x-2.5)^2+(y+11.5)^2=50.5$

4e Midpoint is $\left(\frac{\sqrt{2}+-\sqrt{2}}{2},\frac{4+6}{2}\right)=(0,\ 5)$

radius is $\frac{1}{2}\sqrt{(-\sqrt{2}-\sqrt{2})^2+(6-4)^2}$

$=\frac{1}{2}\sqrt{(-2\sqrt{2})^2+2^2}$

$=\sqrt{3}$

So equation is

$x^2+(y-5)^2=(\sqrt{3})^2$

$x^2+(y-5)^2=3$

4f Midpoint is $\left(\frac{4\sqrt{3}+-2\sqrt{3}}{2},\frac{-\sqrt{3}+(-5\sqrt{3})}{2}\right)$

$=(\sqrt{3},\ -3\sqrt{3})$

Radius is

$\frac{1}{2}\sqrt{(-2\sqrt{3}-4\sqrt{3})^2+(-5\sqrt{3}-(-\sqrt{3}))^2}$

$=\frac{1}{2}\sqrt{(-6\sqrt{3})^2+(-4\sqrt{3})^2}$

$=\sqrt{39}$

So equation is

$(x-\sqrt{3})^2+(y-(-3\sqrt{3}))^2=(\sqrt{39})^2$

$(x-\sqrt{3})^2+(y+3\sqrt{3})^2=39$

5a $(5-3)^2+(3+2)^2$

$=2^2+5^2$

$=4+25$

$=29\neq 5$ so does not lie on the circle

5b $(1-3)^2+(-1+2)^2$
$=(-2)^2+1^2$
$=4+1$
$=5$ so does lie on the circle

5c $(4-3)^2+(3+2)^2$
$=1^2+5^2$
$=1+25$
$=26\neq 5$ so does not lie on the circle

5d $(2-3)^2+(0+2)^2=(-1)^2+2^2$
$=1+4$
$=5$ so does lie on the circle

6a $(-3-5)^2+2^2=(-8)^2+2^2$
$=64+4$
$=68$ so lies on this circle

6b $(-3+2)^2+(2+1)^2$
$=1^2+3^2$
$=1+9$
$=10\neq 8$ so doesn't lie on this circle

6c $(-3-6)^2+(2-2)^2=(-9)^2+0^2$
$=81$ so lies on this circle

7 $(x-1)^2+(y+1)^2=10$

Centre is (1, –1) so gradient of radius to (2, –4) is $\frac{-4--1}{2-1}=\frac{-3}{1}$
$=-3$

Therefore, gradient of tangent is $\frac{1}{3}$ since $\frac{1}{3}\times -3=-1$ and a tangent is perpendicular to a radius

So equation of tangent is

$y+4=\frac{1}{3}(x-2)$
$3y+12=x-2$
$x-3y-14=0$

8 $(x+3)^2+(y+7)^2=34$

Centre is (–3, –7) so gradient of radius to (0, –2) is $\frac{-2--7}{0--3}=\frac{5}{3}$

Therefore, gradient of tangent is $-\frac{3}{5}$ since $\left(-\frac{3}{5}\right)\times\frac{5}{3}=-1$ and a tangent is perpendicular to a radius

So equation of tangent is

$y+2=-\frac{3}{5}(x-0)$
$5y+10=-3x$
$3x+5y+10=0$

9 $x^2+(y-8)^2=153$

Centre is (0, 8) so gradient of radius to (3, –4) is $\frac{-4-8}{3-0}=\frac{-12}{3}$
$=-4$

Therefore, gradient of tangent is $\frac{1}{4}$ since $\frac{1}{4}\times(-4)=-1$ and a tangent is perpendicular to a radius

So equation of tangent is

$y+4=\frac{1}{4}(x-3)$
$y=\frac{1}{4}x-\frac{3}{4}-4$
$y=\frac{1}{4}x-\frac{19}{4}$

10 $(x+4)^2+y^2=20.5$

Centre is (–4, 0) so gradient of radius to (0.5, –0.5) is $\frac{-0.5-0}{0.5--4}=\frac{-0.5}{4.5}$
$=-\frac{1}{9}$

Therefore, gradient of tangent is 9 since $9\times\left(-\frac{1}{9}\right)=-1$ and a tangent is perpendicular to a radius

So equation of tangent is

$y+\frac{1}{2}=9\left(x-\frac{1}{2}\right)$
$y=9x-\frac{9}{2}-\frac{1}{2}$
$y=9x-5$

11a $x^2+y^2=53,\ x+y=5$
$x=5-y$
$(5-y)^2+y^2=53$
$25-10y+y^2+y^2=53$
$2y^2-10y-28=0$
$\Rightarrow y=7, -2$
$x=5-7$
$=-2$
$x=5-(-2)$
$=7$

So they intersect at (–2, 7) and (7, –2)

11b $(x-1)^2+(y+2)^2=17, y+1=0$

$y=-1$

$(x-1)^2+(-1+2)^2=17$

$x^2-2x+1+1=17$

$x^2-2x-15=0$

$\Rightarrow x=-3, 5$

So they intersect at (–3, –1) and (5, –1)

11c $(x-2)^2+(y+1)^2=36,\quad 2x-y+7=0$

$y=2x+7$

$(x-2)^2+(2x+7+1)^2=36$

$(x-2)^2+(2x+8)^2=36$

$x^2-4x+4+4x^2+32x+64=36$

$5x^2+28x+32=0$

$\Rightarrow x=-1.6, -4$

$y=2(-1.6)+7$

$=3.8$

$y=2(-4)+7$

$=-1$

So they intersect at (–1.6, 3.8) and (–4, –1)

11d $y=2x+1,\quad (x+4)^2+(y+6)^2=10$

$(x+4)^2+(2x+1+6)^2=10$

$(x+4)^2+(2x+7)^2=10$

$x^2+8x+16+4x^2+28x+49=10$

$5x^2+36x+55=0$

$\Rightarrow x=-2.2, -5$

$y=2(-2.2)+1$

$=-3.4$

$y=2(-5)+1$

$=-9$

So they intersect at (–2.2, –3.4) and (–5, –9)

12a $3x-9y=6$, $(x+7)^2+(y+3)^2=10$

$3x-9y=6$

$3x=9y+6$

$x=3y+2$

$(3y+2+7)^2+(y+3)^2=10$

$(3y+9)^2+(y+3)^2=10$

$9y^2+54y+81+y^2+6y+9=10$

$10y^2+60y+80=0$

$\Rightarrow y=-2, -4$

$x=3(-2)+2$

$=-4$

$x=3(-4)+2$

$=-10$

So they intersect at A(–4, –2) and B(–10, –4)

12b Length of chord

$AB=\sqrt{(-10--4)^2+(-4--2)^2}$

$=\sqrt{(-6)^2+(-2)^2}$

$=2\sqrt{10}$

13a $2x+4y=10, (x+5)^2+(y-2)^2=20$

$x=5-2y$

$(5-2y+5)^2+(y-2)^2=20$

$(10-2y)^2+(y-2)^2=20$

$100-40y+4y^2+y^2-4y+4=20$

$5y^2-44y+84=0$

$\Rightarrow y=6, 2.8$

$x=5-2(6)$

$=-7$

$x=5-2(2.8)$

$=-0.6$

So they intersect at A(–7, 6) and B(–0.6, 2.8)

13b Length of chord

$AB=\sqrt{(-0.6--7)^2+(2.8-6)^2}$

$=\sqrt{6.4^2+(-3.2)^2}$

$=\frac{16}{5}\sqrt{5}$

14 $(x-3)^2+((x-3)+2)^2=2$

$(x-3)^2+(x-1)^2=2$

$x^2-6x+9+x^2-2x+1=2$

$2x^2-8x+8=0$

$b^2-4ac=(-8)^2-4\times2\times8$

= 0 so only one solution

Hence a tangent

15 $y=34-4x$

$(x+1)^2+((34-4x)-4)^2=68$

$(x+1)^2+(30-4x)^2=68$

$x^2+2x+1+900-240x+16x^2=68$

$17x^2-238x+833=0$

$b^2 - 4ac = (-238)^2 - 4 \times 17 \times 833$
$= 0$ so only one solution

Hence a tangent

16 $x = 25 - 3y$

$(25 - 3y)^2 + (y - 5)^2 = 10$

$625 - 150y + 9y^2 + y^2 - 10y + 25 = 10$

$10y^2 - 160y + 640 = 0$

$b^2 - 4ac = (-160)^2 - 4 \times 10 \times 640$
$= 0$ so only one solution

Hence a tangent

17 $(x - 1)^2 + ((2x + 3) + 4)^2 = 1$

$(x - 1)^2 + (2x + 7)^2 = 1$

$x^2 - 2x + 1 + 4x^2 + 28x + 49 = 1$

$5x^2 + 26x + 49 = 0$

$b^2 - 4ac = 26^2 - 4 \times 5 \times 49$
$= -304$ negative so no solutions

Hence they do not intersect

18 $3x = -2 - 4y$

$x = -\frac{2}{3} - \frac{4}{3}y$

$\left(\left(-\frac{2}{3} - \frac{4}{3}y\right) + 3\right)^2 + (y - 6)^2 = 9$

$\left(\frac{7}{3} - \frac{4}{3}y\right)^2 + (y - 6)^2 = 9$

$\frac{49}{9} - \frac{56}{9}y + \frac{16}{9}y^2 + y^2 - 12y + 36 = 9$

$\frac{25}{9}y^2 - \frac{164}{9}y + \frac{292}{9} = 0$

$b^2 - 4ac = \left(-\frac{164}{9}\right)^2 - 4 \times \frac{25}{9} \times \frac{292}{9}$

$= -\frac{256}{9}$

$\frac{-256}{9} < 0$ so no solutions

Hence they do not intersect